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FULLY SMALL STABLE S-SYSTEMS

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Abstract. This study aims to provide an overview and research into the fully small stable systems as a concept that encompasses fully small stable modules as well as fully stable systems but is more potent than duo systems. We look at some of the characteristics and characterizations of the class of fully small stable systems, and the relationships between it and other types. Among these categories are fully stable systems, systems that fulfill Baer's criterion, quasi-injective systems, small duo systems, and small principally quasi injective systems. A fully small stable system does not need to be fully stable in general but coincides when the radical of an S-system is equal to that of S-system. Also, a fully stable small system is equivalent to an S-system that satisfies Baer's criterion for small cyclic subsystems. The system, which is fully small stable is identical to the duo system, which is also a small principally quasi-injective system.

Keywords: small subsystem, S-acts, radical, Small duo, fully stable, principally quasi injective, Baer's criterion, S-system, quasi-injective.

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1 Introduction

The action of a semigroup or a monoid on a set is a fundamental concept in several fields of mathematics and some branches of new technologies. A right S-system over a monoid S is a set B together with the function $B \times S \to B$, $(b, r) \to br$, such that $b1 = b$ and $(br)t = b(rt)$ for all $b \in B$, $r, t \in S$. A right S-system may thus be thought of as a natural generalization of right modules over rings. Many of the problems posed and answered in module theory can also be asked and answered in S-system, although the answers are frequently different.

The main distinction is that an S-system has no underlying group structure, which means that special subsets in general cannot establish congruences. The reader may refer to Kilp et al. (2000) and Clifford & Preston (1961, 1967) for additional information concerning S-systems.

Abbas & Salman (2018) developed a class of modules that is named fully small stable modules as the following:
If $\theta(B) \subseteq B$ for each small submodule B of A and each R-homomorphism $\theta : B \to A$, a right R-module A which we shall refer to as fully small stable module. We present fully small stable systems (which we abbreviate as FSS) as a new class of systems and provide many characterizations of FSS systems in this work. The following definition will be our starting point:

Definition 1. *Let B be a subsystem of the S-system A. B is called a small in A and which denoted by $B \leqslant_s A$, when each subsystem C of A with $B \cup C = A$, implies $C = A$ (Abdul-kareem, 2021).*

We now present some results which may be found in Abdul-kareem (2021, 2020).

Lemma 1. *If K_j is a small subsystem of the S-system A for all $j = 1, 2, ..., m$, then $\cup_{j=1}^{m} K_j \leqslant_s A$.*

Lemma 2. *Let A be a S-system, $C \subseteq K \subseteq A$ and $K \leqslant_s A$, then $C \leqslant_s A$.*

Definition 2. *A small subsystem B of the right S-system A_S is called stable, if $\Theta(B) \subseteq B$ for all S-homomorphisms $\Theta : B \to A$. A_S is referred to as fully small stable (which we abbreviate as FSS), if all small subsystems of A_S are stable. A monoid S is FSS, if it is FSS as a S-system.*

Recall that the radical of the S-system A is the union of all small subsystems of the S-system A, represented by $Rad(A)$ Abdul-kareem (2020).

Lemma 3. *Let A_S be a S-system. Then, $a \in Rad(A)$ if and only if $aS \leqslant_s A$.*

Proof. Let $a \in Rad(A)$. Hence, there exists $A_0 \subseteq Rad(A)$ with $a \in A_0 \leqslant_s A$, since $aS \subseteq A_0$. So that, $aS \leqslant_s A$. The proof of the converse is clear. □

The following proposition shows that to determine whether a system is a FSS, it suffices to consider the stability of a very restricted class of small subsystems.

Proposition 1. *A S-system A_S is FSS if and only if, each small cyclic subsystem of A_S is stable.*

Proof. Let Δ be a S-homomorphism from a small subsystem B_S into A_S. Hence, for all b in B, $bS \subseteq B$. *Lemma 3*, assures that $bS \leqslant_s A$. Thus, $\Delta(bS) \subseteq bS$, which assures that $\Delta(B) \subseteq B$. □

Examples and Remarks 1.

a. *Let A be a S-system :*

   1. *$A \leqslant_s A$ if and only if $A = 0$.*
   2. *$0 \leqslant_s A$*

b. *Recall that a right S-system B_S is fully stable system, if $\theta(A) \subseteq A$ for each subsystem A of B_S and each S-homomorphism $\theta : A \to B$ (Abbas & Baanoon (2015)). Every fully stable system is a FSS. However, the converse is not always true. As an example, a semigroup $S = \{0, x, y\}$ with $xy = x^2 = x$ and $yx = y^2 = y$ as an S-system is FSS system since $0S$ is the only small subsystem of S_S that is stable. While there is a S-homomorphism $\delta : \{0, x\} \to S$ defined by $\delta(0) = 0$ and $\delta(x) = y$, clearly $\delta(\{0, x\}) \nsubseteq \{0, x\}$. Hence, S_S is not a fully stable system.*

c. *If $Rad(A) = A$ where A be a S-system, fully stable systems and FSS systems are identical.*

d. *Every proper subsystem of $\mathbb{Z}$ as a $(\mathbb{Z}, .)$-system is small.*

   Proof. Let $n\mathbb{Z}$ be a proper subsystem of $\mathbb{Z}$, which assures that $n\mathbb{Z} \neq \mathbb{Z}$. Hence $n \neq 1$ with $n\mathbb{Z} \cup A = \mathbb{Z}$, for some subsystem A of $\mathbb{Z}$ and since $1 \in \mathbb{Z}$, so that $1 \in A$. Hence $A = \mathbb{Z}$. □

e. *$\mathbb{Z}$ as a $(\mathbb{Z}, .)$-system is not FSS. Because there exists an S-homomorphism $\alpha : 3\mathbb{Z} \to \mathbb{Z}$ it is defined by $\alpha(3z) = 2z$ for all $z \in 3\mathbb{Z}$. Obviously, $\alpha(3\mathbb{Z}) \nsubseteq 3\mathbb{Z}$, where $3\mathbb{Z}$ is a small.*

f. *Let A be a S-system, $A_1, A_2, ..., A_n$ be small stable subsystems of A. Then $\cup_{i=1}^{n} A_i$ is small stable subsystem. Because the union of stable subsystems is stable Baanoon (2014), also from Lemma 2, the finite union of small subsystems is small. The finite union of small stable subsystems is a small stable subsystem.*

We now present another characterization of FSS systems.

Proposition 2. *Assume A_S is a S-system. Then the following are equivalent:*

i. *A_S is FSS.*

ii. *Each subsystem B of A_S is FSS.*

iii. *If C and B are two subsystems of A_S such that $B \leqslant_s A$ and C is an epimorphic image of B, then $C \subseteq B$*

Proof. $(i) \Leftrightarrow (ii)$ The proof is straightforward.
$(i) \Rightarrow (iii)$ Suppose that C is a subsystem of A and $B \leqslant_s A$ with the S-epimorphism $\delta : B \to C$. Hence for all $c \in C$ there exists $b \in B$ such that $\delta(b) = c$. Consider the inclusion S-homomorphism $i : C \to A$. Since $B \leqslant_s A$ and A is FSS system, then $(i \circ \delta)(B) \subseteq B$. Hence $\delta(B) \subseteq B$. But $\delta(B) = C$. Hence, $C \subseteq B$.
$(iii) \Rightarrow (i)$ Suppose that B is a small subsystem of A and $\delta : B \to A$ is a homomorphism. Thus, $\delta : B \to \delta(B)$ is a epimorphism. By setting $\delta(B) = C$, our assumption assures that $\delta(B) \subseteq B$. □

Regarding the annihilators of its small cyclic subsystem, the following statement characterizes FSS systems, where $\mathcal{R}_S(a) = \{(t_1, t_2) \in S \times S : at_1 = at_2\}$ and $\mathcal{R}_S(b) = \{(t_1, t_2) \in S \times S : bt_1 = bt_2\}$ and $\mathcal{L}_A(\mathcal{R}_S(a)) = \{c \in A : ct_1 = ct_2, \forall (t_1, t_2) \in \mathcal{R}_S(a)\}$, where A be a S-system (Kim, 2008).

Proposition 3. *Let A be an S-system, where S is a monoid and commutative.*

i. *A is a FSS system.*

ii. *$\mathcal{L}_A(\mathcal{R}_S(a)) = aS$ for each a in $Rad(A)$.*

iii. *$\mathcal{R}_S(a) \subseteq \mathcal{R}_S(b)$ implies that $b \in aS$ for each $a \in Rad(A)$ and each $b \in A$.*

Proof. $(i) \Rightarrow (ii)$ Let a in $Rad(A)$. $aS \subseteq \mathcal{L}_A(\mathcal{R}_S(a))$, since let $x \in aS$. Thus, for each $(t_1, t_2) \in \mathcal{R}_S(a)$, $xt_1 = ayt_1 = at_1y = at_2y = ayt_2 = xt_2$ for some $y \in S$. Let $b \in \mathcal{L}_A(\mathcal{R}_S(a))$ where $\alpha : aS \to A$ is a S-homomorphism which is defined by $\alpha(ar) = br$ for each $r \in S$. Hence, by (i) we have that $\alpha(aS) \subseteq aS$. Thus, $b \in aS$. Therefore $\mathcal{L}_A(\mathcal{R}_S(a)) = aS$ for each a in $Rad(A)$.
$(ii) \Rightarrow (iii)$ Let $\mathcal{R}_S(a) \subseteq \mathcal{R}_S(b)$. Then $bS \subseteq \mathcal{L}_A(\mathcal{R}_S(b))$ and from (ii) we have that $aS = \mathcal{L}_A(\mathcal{R}_S(a))$. $\mathcal{R}_S(a) \subseteq \mathcal{R}_S(b)$ implies that $\mathcal{L}_A(\mathcal{R}_S(b)) \subseteq \mathcal{L}_A(\mathcal{R}_S(a))$. Hence, $bS \subseteq \mathcal{L}_A(\mathcal{R}_S(b)) \subseteq \mathcal{L}_A(\mathcal{R}_S(a)) = aS$. Therefore, $b \in aS$.
$(iii) \Rightarrow (i)$ Suppose that aS is a small cyclic subsystem of A and $\alpha : aS \to A$ is an S-homomorphism. Then $\mathcal{R}_S(a) \subseteq \mathcal{R}_S(\alpha(a))$ and by (iii) we have that $\alpha(a)$ in aS. Thus, $\alpha(aS) \subseteq aS$. □

2 Fully small stable system and Baer criterion

Remember that a subsystem B_S of a system A_S fulfills Baer's criterion if there exists an element $t \in S$ such that $\theta(b) = bt$ for all $b \in B_S$ for every S-homomorphism $\theta : B_S \to A_S$. If every subsystem of a S-system A_S fulfills Baer's criterion, the system is said to satisfy Baer's criterion (Abbas & Baanoon, 2015).

Proposition 4. *Assume A_S is a system, where S is a monoid and commutative. Then, A_S is FSS if and only if A_S satisfies Baer criterion on its small cyclic subsystem.*

Proof. $\Rightarrow$ Assume that aS is a small cyclic subsystem of A_S and $\beta : aS \to A$. Since A is FSS. Thus, $\beta(aS) \subseteq aS$. Hence, for each $b \in aS$ there exists $r \in S$ such that $\beta(a) = ar$. Let $q \in aS$. Thus, $q = as$ for some $s \in S$ and hence $\beta(q) \in aS$. So, $\beta(q) = \beta(as) = \beta(a)s = (ar)s = x(tr) = a(rs) = (as)r = qr$. Thus, there is $r \in S$, $\ni \beta(q) = qr$ for each $q \in aS$. Hence, the Baer criterion is satisfied in small cyclic subsystems.
$\Leftarrow$ Assume B is a small cyclic subsystem of A and $\theta : B \to A$ is a S-homomorphism. For each $b \in B$, $bS \leqslant_S A$. So, from the hypothesis there is $t \in S$ such that $\theta(b) = bt$. Hence, $\theta(bS) \subseteq bS$ for each $b \in B$. In particular $\theta(b) \in bS \subseteq B$. Then $\theta(B) \subseteq B$. Therefore, A is FSS. □

3 Fully small stable and quasi-injective systems

Remember that a system A_S is considered quasi-injective, if there exists a S-homomorphism $\psi : A_S \to A_S$ extending φ for any subsystem B_S of A_S and any S-homomorphism $\varphi : B_S \to A_S$ Kilp et al. (2000). Recall that a S-system A is a duo system if each subsystem B of A is fully invariant i.e., $\phi(B) \subseteq B$ for each S-endomorphism ϕ of A see Roueentan & Ershad (2014). Under the condition that S is a monoid and commutative, we shall examine the relationship between small duo and FSS systems.
We shall now introduce the small duo system.

Definition 3. *A system A_S is said the small duo if each small subsystem of A_S is fully invariant. S is a right small duo if S_S is a small duo.*

Remark 1.

i. *Every FSS is a small duo S-system.*

   Proof. Assume A_S is a FSS system and B is a small subsystem of A_S with an S-endomomorphism $\alpha : A \to A$. Hence, $\alpha|_B : B \to A$ is a S-homomorphism. Thus, $\alpha(B) \subseteq B$. Therefore, A is a small duo system. □

ii. *Small duo S-system need not be FSS in general.*

iii. *Every duo system is the small duo.*

Proposition 5. *Assume A_S is a quasi-injective system. Then the following are equivalent.*

i. *A is FSS system.*

ii. *A is a small duo system.*

Proof. $(i) \Rightarrow (ii)$ The proof is straightforward.
$(ii) \Rightarrow (i)$ Let B be a small subsystem of A and $\delta : B \to A$ be an S-homomorphism. Since A is quasi-injective, there exists a S-endomorphism $\Delta : A \to A$ such that $\delta(y) = \Delta(y)$ for all $y \in B$. A is small duo. Thus, $\Delta(B) \subseteq B$. Hence, $\delta(B) = \Delta(B) \subseteq B$. Therefore, A is a FSS system. □

Remember that a system B_S is a small principally A-injective if any S-homomorphism from a small cyclic subsystem of a system A_S into the system B_S can be extended to a S-homomorphism from A_S into B_S. If A_S is small principally A-injective, A_S is called a small principally quasi injective (which we abbreviate as SPQ-injective) (Abdul-kareem, 2021).

Proposition 6. *Assume A_S is a system where S is commutative monoid. Then:*

i. *A_S is duo and SPQ-injective.*

ii. A_S *is a* FSS *system.*

Proof. $(i) \Rightarrow (ii)$ Let B_S be a small cyclic subsystem of A_S and $\delta : B \to A$ be a S-homomorphism, from the hypothesis as, A_S is SPQ-injective.
Thus, there exists an extended endomorphism $\Delta : A \to A$ of δ, since A_S is a small duo. Hence, $\delta(B) = \Delta(B) \subseteq B$. Therefore, A_S is a FSS system.

$(ii) \Rightarrow (i)$ It is sufficient to show that every FSS system is SPQ-injective. Let $a \in Rad(A)$ and $\delta : aS \to A$ be an S-homomorphism. Since A is FSS, there is a $s \in S$ such that $\delta(a) = as$. We define $\Delta : A \to A$ as $\Delta(x) = xs$ for all $x \in A$. Hence, Δ is an extended endomorphism for δ. Therefore, A is SPQ-injective.

□

4 Conclusion

This paper defines the fully small stable system, gives examples, finds other characterizations, and proves some properties of that concept. Also, we define a small duo system, and we see that when a system is a quasi-injective system, then the fully small stable system and small duo system are equivalent. Also, we see that when a system on commutative monoid is a duo, then the fully small stable system and small principally quasi injective are identical.

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